International Journal of Theoretical Physics, Vol. 36, No. 12, 1997

Quantum Mechanics: From Complex to Complexified Quaternions

Stefano De Leo[1,2] and Waldyr A. Rodrigues, Jr.[2]

Received May 5, 1997

This paper is an attempt to simplify and clarify the mathematical language used to express quaternionic quantum mechanics (QQM). In our quaternionic approach the choice of "complex" geometries allows an appropriate definition of momentum operator and gives the possibility to obtain consistent formulations of standard theories. Barred operators represent the key to realizing a set of translation rules between quaternionic and complex quantum mechanics (QM). These translations enable us to obtain a rapid quaternionic counterpart of standard quantum mechanical results.

1. INTRODUCTION

Since the discovery of quarternions by Sir William Rovan Hamilton (1943, 1969) a recurring question has been posed: Is it possible to formulate quaternionic physical theories? After the fundamental works of Finkelstein *et al.* (1962, 1963a, b, 1979) on quaternionic versions of gauge theories and QM, in recent years there has been renewed interest in physical applications for noncommutative fields. Among the numerous references on this subject, we recall the important paper of Horwitz and Biedenharn (1984), where the authors showed that the assumption of a complex scalar product, "complex" geometry (Rembieliński, 1978) permits the definition of tensor products between single-particle wave functions, without encountering intractable problems of interpretation and definition due to the noncommuting multiplications of quaternionic wave functions. We also mention the recent book of

[1] Dipartimento di Fisica, Università degli Studi Lecce and INFN, Sezione di Lecce, CP 193, 73100 Lecce, Italia; e-mail: deleos@le.infn.it and deleo@ime.unicamp.br.
[2] Instituto de Matemática, Estatística e Computação Científica, IMECC-UNICAMP CP 6065, 13081-970, Campinas, S. P., Brazil; e-mail: walrod@ime.unicamp.br.

0020-7748/97/1200-2725$12.50/0

Adler (1995), where a complete and clear explanation of QQM with quaternionic geometry is formulated. In his book, Adler also suggested the stimulating possibility to look at the color degree of freedom in the Harari–Shupe model (Harari, 1979; Shupe, 1979) by a *noncommutative* generalization (Adler, 1994) of standard (complex) QM.

In this work we propose a simple quaternionic language for the formulation of physical theories. We also exhibit explicitly a set of rules for passing back and forth between standard QM and its quaternionic versions. The possibility to unify the different formulation of QM, namely

$$\text{complex} \leftrightarrow \text{real quaternionic} \leftrightarrow \text{complexified quaternionic}$$

suggests that the standard (complex) version of physical theories represents a choice and not a necessity. We explicitly show that the different formulations of the Dirac equation by complex, real, and complexified quaternions have the same physical content. In fact, by our translation rules we shall connect 4×4 complex matrices with two-dimensional *real* and one-dimensional *complexified* quaternionic operators. The extension of complex fields to quaternionic structures gives special advantages. We like calling our translations "partial translations" since the quaternionic approach provides additional physical predictions and new geometric interpretations. We shall see that the use of complexified quaternions and their possible translation by 2×2 complex matrices has the interesting result of giving a two-dimensional complex representation of the Dirac equation.

The paper is structured as follows. After a brief introduction to quaternionic algebras, we explain the concept of *barred operators* (Section 2). In Section 3 we discuss the choice of "complex" geometries and obtain an appropriate definition of momentum operator within QQM. In Sections 4 and 6 we give the main tools to perform the translation rules, while in Section 5 we present an interesting example of translation between complex and quaternionic QM, the Dirac equation. Our conclusions are drawn in the final section.

2. QUATERNIONIC ALGEBRAS

Complex numbers can be constructed from the real numbers by introducing a quantity i whose square is -1:

$$z = r_1 + ir_2, \qquad r_{1,2} \in \mathcal{R}$$

Likewise, we can construct the quaternions from the complex numbers in exactly the same way by introducing another quantity j whose square is -1,

$$q = z_1 + jz_2, \qquad z_{1,2} \in C(1, i)$$

and which anticommutes with i ($ij = -ji = k$). We wish to emphasize the need for *three anticommuting* imaginary units in constructing the quaternionic field (only two imaginary units are not sufficient to obtain the Hamilton field).

In 1843, Hamilton attempted to generalize the complex field in order to describe the rotations in three-dimensional space. He began by looking for numbers of the form $x + iy + jz$, with $i^2 = j^2 = -1$. Hamilton's hope was to do for three-dimensional space what complex numbers do for the plane. Influenced by the existence of a complex number norm

$$z^*z = (\text{Re } z)^2 + (\text{Im } z)^2$$

when he looked at its generalization

$$(x - iy - jz)(x + iy + jz) = x^2 + y^2 + z^2 - (ij + ji)yz$$

to obtain a real number, he had to adopt the anticommutative law of multiplication for the imaginary units. Nevertheless, as remarked before, with only two imaginary units we have no chance of constructing a new numerical field, because assuming

$$ij = \alpha_0 + i\alpha_1 + j\alpha_2 \qquad (\alpha_{0,1,2} \in \mathcal{R})$$

$$ji = \beta_0 + i\beta_1 + j\beta_2 \qquad (\beta_{0,1,2} \in \mathcal{R})$$

and

$$ij = -ji$$

we find the relation $\alpha_{0,1,2} = \beta_{0,1,2} = 0$. Thus, we must introduce a third imaginary unit $k \neq i/j$, with

$$k = ij = -ji$$

2.1. Real Quaternionic Algebra

The new noncommutative field is characterized by three imaginary units i, j, k which satisfy the following multiplication rules:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

Equation (1) contains the solution of the problem which "haunted" Hamilton for at least 15 years (Appendix A1). Numbers of the form

$$q = x_0 + ix + jy + kz \qquad (x_0, x, y, z \in \mathcal{R}) \tag{2}$$

are called (real) *quaternions*. They are added, subtracted, and multiplied according to the usual laws of arithmetic, except for the commutative law of multiplication.

Similarly to rotations in a plane that can be concisely expressed by complex number, a rotation about an axis passing through the origin and parallel to a given unitary vector $\hat{u} \equiv (u_x, u_y, u_z)$ by an angle α can be obtained taking the following *quaternionic* transformation:

$$\exp\left(\frac{\alpha}{2}\,\mathbf{h}\cdot\mathbf{u}\right)\mathbf{h}\cdot\mathbf{r}\,\exp\left(-\frac{\alpha}{2}\,\mathbf{h}\cdot\mathbf{u}\right)$$

where $\mathbf{h} \equiv (i, j, k)$, $\mathbf{r} \equiv (x, y, z)$, and $\mathbf{h}\cdot\mathbf{r} = ix + jy + kz$. A similar approach to rotations was introduced by Olinde Rodrigues in 1840, before Hamilton's discovery of quaternions (Appendix A2).

We recall that the quaternion q in equation (2), with the identification $x_0 \equiv ct$, can be used to formulate a one-dimensional version of the Lorentz group (De Leo, 1996b). This gives the natural generalization of Hamilton's idea

complex/plane $\to$ pure imaginary

quaternions/space $\to$ quaternions/space-time

completing the unification of algebra and geometry.

To conclude this introduction to the real quaternionic algebra, we introduce the quaternionic conjugation operation denoted by † and defined by

$$q^\dagger = x_0 - ix - jy - kz \tag{3}$$

We note that the previous conjugation implies

$$(qp)^\dagger = p^\dagger q^\dagger$$

We also observe that $q^\dagger q$ and $qq^\dagger$ are both equal to the real number

$$N(q) = x_0^2 + x^2 + y^2 + z^2$$

which is called the norm of q. When $q \neq 0$, we can define $q^{-1} = q^\dagger/N(q)$, so the quaternions form a zero-division ring. Such a noncommutative number field is denoted, in Hamilton's honor, by $\mathcal{H}$.

Whereas with complex numbers we can define only one type of conjugation $\iota \to -\iota$ (we use the notation *iota* to distinguish the imaginary complex unit ι from the imaginary quaternionic unit i), working with quaternionic numbers, we can introduce different conjugation operations. Indeed, with three imaginary units we have the possibility to define, besides the standard conjugation (3), the six new operations

$$(i, j, k) \to (-i, +j, +k), (+i, -j, +k), (+i, +j, -k)$$

$$(i, j, k) \to (+i, -j, -k), (-i, +j, -k), (-i, -j, +k)$$

Nevertheless, the previous six conjugations are not independent. In fact, we can prove that the conjugation operations which change only one imaginary unit are connected between themselves and with the conjugation of all three imaginary units through similarity transformations, as in

$$q \to -iq^{\dagger}i,\ -jq^{\dagger}j,\ -kq^{\dagger}k$$

An analogous observation can be formulated for the conjugation of two imaginary units

$$q \to -iqi,\ -jqj,\ -kqk$$

2.1.1. *Real Linear Barred Operators*

Due to the noncommutative nature of quaternions, we must distinguish between q_1q_2 and q_2q_1. Thus, it is appropriate to consider left/right-actions for our imaginary units i, j, and k. We introduce *barred operators* to represent, in a compact way, the right-action of the three quaternionic imaginary units. Explicitly, we write

$$1|i, \quad 1|j, \quad 1|k \tag{4}$$

to identify the right multiplication of i, j, k and so

$$(1|\mathbf{h})q \equiv q\mathbf{h}$$

where $\mathbf{h} \equiv (i, j, k)$. In this formalism, the most general transformation on quaternions will be given by

$$q_0 + q_1|i + q_2|j + q_3|k \qquad (q_{0,1,2,3} \in \mathcal{H}) \tag{5}$$

In the last few years the left/right-actions of the quaternionic numbers, expressed by barred operators (5), have been very useful in overcoming difficulties owing to the noncommutativity of quaternions. Among the successful applications of barred operators we mention the one-dimensional quaternionic formulation of Lorentz boosts. Explicitly, the quaternionic generators of the Lorentz group are

boost (ct, x)	$\frac{k\|j - j\|k}{2}$
boost (ct, y)	$\frac{i\|k - k\|i}{2}$
boost (ct, z)	$\frac{j\|i - i\|j}{2}$
rotation around x	$\frac{i - 1\|i}{2}$
rotation around y	$\frac{j - 1\|j}{2}$
rotation around z	$\frac{k - 1\|k}{2}$

The four real quantities which identify the space-time point (ct, x, y, z) are represented by the quaternion

$$q = ct + ix + jy + kz$$

The use of barred operators (5) gives us the possibility to extend the connection between the special unitary group $SU(2)$ and the unitary quaternions by allowing a *one-dimensional* quaternionic version of the special linear group $SL(2)$ (De Leo, 1996b).

Remembering the noncommutativity of the quaternionic multiplication, we must specify if our scalar factors are quaternionic, complex, or real numbers. Operators which act on states *only* from the left (i.e., quaternionic numbers) will be named *quaternionic linear operators* and will be simply indicated by q. Obviously, from these, more general classes of operators, such as complex or real linear quarternionic operators, can be constructed. For example, the barred operator (5) represents a *real linear* quaternionic operator. To complete the list of possible barred operators we give an explicit example of *complex linear* quaternionic operator

$$\mathbb{O}^i \equiv q_0 + q_1 | i \tag{6}$$

2.1.2. *Complex Linear Barred Operators*

Let us now discuss the algebra of complex linear barred operators and introduce some elementary relations and definitions which will be useful in the following sections. The product of two complex linear barred operators $\mathbb{O}^i_q$ and $\mathbb{O}^i_p$ in terms of quaternions $q_{0,1}$ and $p_{0,1}$ is given by

$$\mathbb{O}^i_q \mathbb{O}^i_p = q_0 p_0 - q_1 p_1 + (q_0 p_1 + q_1 p_0) | i$$

The "full" conjugation operation is defined by changing the sign of our left/right quaternionic imaginary units, i.e.,

$$(i, j, k)^\dagger = -(i, j, k) \qquad \text{and} \qquad (1|i)^\dagger = -1|i$$

The previous definition implies

$$(\mathbb{O}^i_q \mathbb{O}^i_p)^\dagger = \mathbb{O}^{i\,\dagger}_p \mathbb{O}^{i\,\dagger}_q$$

We observe that complex linear operators (6) are characterized by four complex numbers and this suggest a possible connection between complex linear quaternionic operators and 2×2 complex matrices (De Leo and Rotelli, 1994).

2.2. Complexified Quaternionic Algebra

Up to now we have been working with a quaternionic algebra over a real field. Obviously, we can define a more general quaternionic algebra over a field $\mathcal{F} \neq \mathcal{R}$; in particular, we can introduce the so-called complexified quaternionic algebra

$$\mathcal{H}_c = \{c_0 + ic_1 + jc_2 + kc_3 \; c_{0,1,2,3,} \in C(1, \iota)\} \tag{7}$$

where the new imaginary unit ι ($\neq i$) commute with the quaternionic imaginary units i, j, k

$$[\iota, \mathbf{h}] = 0$$

Working with complexified quaternions, we have three different (independent) opportunities to define conjugation operations

$$q_c^\bullet = c_0^* + \mathbf{h} \cdot \mathbf{c}^*$$
$$q_c^\star = c_0 - \mathbf{h} \cdot \mathbf{c}$$
$$q_c^\dagger = c_0^* - \mathbf{h} \cdot \mathbf{c}^*$$

where $*$ indicates the standard complex conjugation ($\iota \to -\iota$). It is straightforward to prove that

$$(q_c p_c)^\bullet = q_c^\bullet p_c^\bullet$$
$$(q_c p_c)^\star = p_c^\star q_c^\star$$

and consequently

$$(q_c p_c)^\dagger = (q_c p_c)^{\bullet\star} = (q_c^\bullet p_c^\bullet)^\star = p_c^{\bullet\star} q_c^{\bullet\star} = p_c^\dagger q_c^\dagger$$

By introducing barred operators, we must admit two kinds of complex linear barred operators. In fact, we can have ι-complex linearity or i-complex linearity (note that ι refers to the complex field $\mathcal{C}$, whereas i refers to the quaternionic imaginary units of $\mathcal{H}$).

2.2.1. ι-Complex Linear Barred Operators

Admitting ι-complex linearity, the most general transformation on complexified quaternions will be given by

$$\mathcal{O}_c^\iota \equiv q_c + p_c|i + r_c|j + s_c|k \qquad (q_c, p_c, r_c, s_c \in \mathcal{H}_c) \tag{8}$$

The product of two ι-complex barred operators $\mathcal{O}_c^{\iota,1}$ and $\mathcal{O}_c^{\iota,2}$ in terms of complexified quaternions is given by

$$\begin{aligned}\mathcal{O}_{c,1}^\iota \mathcal{O}_{c,2}^\iota = {} & q_{c,1}q_{c,2} - p_{c,1}p_{c,2} - r_{c,1}r_{c,1} - s_{c,1}s_{c,1} \\ & + (q_{c,1}p_{c,2} + p_{c,1}q_{c,2} - r_{c,1}s_{c,2} + s_{c,1}r_{c,2})|i \\ & + (q_{c,1}r_{c,2} + r_{c,1}q_{c,2} - s_{c,1}p_{c,2} + p_{c,1}s_{c,2})|j \\ & + (q_{c,1}s_{c,2} + s_{c,1}q_{c,2} - p_{c,1}r_{c,2} + r_{c,1}p_{c,2})|k\end{aligned}$$

The conjugation operations are defined as follows:

$$\mathbb{O}_c^{\iota\bullet} \equiv q_c^{\bullet} + p_c^{\bullet}|i + r_c^{\bullet}|j + s_c^{\bullet}|k$$

$$\mathbb{O}_c^{\iota\star} \equiv q_c^{\star} - p_c^{\star}|i - r_c^{\star}|j - s_c^{\star}|k$$

$$\mathbb{O}_c^{\iota\dagger} \equiv q_c^{\dagger} - p_c^{\dagger}|i - r_c^{\dagger}|j - s_c^{\dagger}|k$$

The $\bullet$ involution represents an automorphism, while the $\star$ and † conjugations are antiautomorphisms. If we analyze the transformation (8), we immediately note that such a transformation is characterized by 16 ι-complex parameters and this can be used to connect ι-complex linear barred operators to 4×4 complex matrices (De Leo, 1996a). The analogy will be completed when will introduce an ι-complex geometry which guarantees the orthogonality of the following complexified quaternionic fields:

$$c, \quad ic, \quad jc, \quad kc \qquad c \in \mathcal{C}(1, \iota)$$

and consequently allows the identification of a complexified quaternionic state by a four-dimensional complex vector column.

2.2.2. ι-Complex Linear Barred Operators

As previously remarked, we must admit two kinds of complex linear barred operators. Together with ι-complex linearity (just discussed), we have to introduce i-complex linearity. In this last case, the most general transformation on complexified quaternions will be given by

$$\mathbb{O}_c^{i} \equiv q_c + p_c|i \qquad (q_c, p_c \in \mathcal{H}_c) \tag{9}$$

Noting that

$$\mathbb{O}_c^{i} \subset \mathbb{O}_c^{\iota}$$

we can obtain the algebra of i-complex linear barred operators directly from the one of ι-complex linear barred operators, killing the $1 \mid j$ and $1 \mid k$ terms. We conclude this brief discussion by some considerations on the *complex freedom* degrees of the transformation (9). The barred operator $\mathbb{O}_c^{i}$ is characterized by eight i-complex numbers

$$\mathbb{O}_c^{i} \equiv z_1 + j\tilde{z}_1 + \iota(z_2 + j\tilde{z}_2) + [z_3 + j\tilde{z}_3 + \iota(z_4 + j\tilde{z}_4)] \mid i,$$

$$z_{1,2,3,4}, \tilde{z}_{1,2,3,4} \in C(1, i) \subset \mathcal{H}$$

Apparently, we have no possibilities of relating such operators to 4×4 complex matrices, and consequently to express the Dirac algebra. Nevertheless, the $\bullet$ involution will enable us to obtain the *missing* complex freedom degrees, providing the desired (unexpected) translation.

3. "COMPLEX" GEOMETRIES

The noncommutativity of the quaternionic multiplication requires that we specify whether the quaternionic Hilbert space is to be formed by right or left multiplication of vectors by scalars. We must also specify whether our scalars are quaternionic, complex, or real numbers. We will follow the usual choice (Horwitz and Biedenharn, 1984; Adler, 1995) and work with a linear vector space under right multiplication by scalars.

In QM, probability amplitudes, rather than probabilities, superimpose, so we must determine what kinds of number system can be used for the probability amplitudes $\mathcal{A}$. We need a real modulus function $N(\mathcal{A})$ such that

$$\text{Probability} = [N(\mathcal{A})]^2$$

The first four assumptions on the modulus function are basically technical in nature

$$N(0) = 0$$

$$N(\mathcal{A}) > 0 \qquad \text{if} \quad \mathcal{A} \neq 0$$

$$N(r\mathcal{A}) = |r|N(\mathcal{A}), \qquad r \text{ real}$$

$$N(\mathcal{A}_1 + \mathcal{A}_2) \leq N(\mathcal{A}_1) + N(\mathcal{A}_2)$$

A final assumption about $N(\mathcal{A})$ is physically motived by imposing the *correspondence principle* in the following form: We require that in the absence of quantum interferences effects, probability amplitude superimposition should reduce to probability superimposition. So we have an additional condition on $N(\mathcal{A})$:

$$N(\mathcal{A}_1\mathcal{A}_2) = N(\mathcal{A}_1)N(\mathcal{A}_2)$$

A remarkable theorem of Albert (1947) shows that the only algebras over the reals admitting a modulus function with the previous properties are the reals $\mathcal{R}$, the complex $\mathcal{C}$, the (real) quaternions $\mathcal{H}$, and the octonions $\mathbb{O}$. The previous properties of the modulus function seem to constrain us to work with *division algebras* (which are finite-dimensional algebras for which $a \neq 0$, $b \neq 0$ imply $ab \neq 0$), in fact

$$\mathcal{A}_1 \neq 0, \qquad \mathcal{A}_2 \neq 0$$

implies

$$N(\mathcal{A}_1\mathcal{A}_2) = N(\mathcal{A}_1)N(\mathcal{A}_2) \neq 0$$

which gives

$$\mathcal{A}_1\mathcal{A}_2 \neq 0$$

A simple example of a nondivision algebra is provided by the algebra of complexified quaternions since

$$(1 + i\iota)(1 - i\iota) = 0$$

guarantees that there are nonzero divisors of zero. So, if the probability amplitudes are assumed to be complexified quaternions, we cannot give a satisfactory probability interpretation. Nevertheless, we know that probability amplitudes are connected to inner products, and thus, we can overcome the above difficulty by defining an *appropriate* scalar product.

3.1. Real QQM

Within real QQM we can define three scalar products. We will call the binary mapping $\langle\psi|\varphi\rangle$ of $V_{\mathcal{H}} \times V_{\mathcal{H}}$ into $\mathcal{H}$, defined by

$$\langle\psi|\varphi\rangle = \int d^3x\, \psi^\dagger\varphi \tag{10}$$

the quaternionic scalar product and the binary mapping $\langle\psi|\varphi\rangle_c$ of $V_{\mathcal{H}} \times V_{\mathcal{H}}$ into $C(1, i)$, defined by

$$\langle\psi|\varphi\rangle_c = \frac{1 - i|i}{2}\langle\psi|\varphi\rangle \tag{11}$$

the complex scalar product or complex geometry. The last trivial possibility is represented by a real scalar product, the binary mapping $\langle\psi|\varphi\rangle_r$ of $V_{\mathcal{H}} \times V_{\mathcal{H}}$ into $\mathcal{R}$, defined by

$$\langle\psi|\varphi\rangle_r = \frac{1 - i|i - j|j - k|k}{4}\langle\psi|\varphi\rangle \tag{12}$$

In real QQM we use a linear quaternionic Hilbert space under right multiplication by complex scalars and work with complex scalar products.

3.1.1. "Complex" Momentum Operator

We justify the choice of a complex geometry by recalling that although there is in QQM an anti-self-adjoint operator $\boldsymbol{\partial}$ with all the properties of a translation operation, imposing a quaternionic geometry, there is *no* corresponding quaternionic self-adjoint operator with all the properties expected for a momentum operator. This "hopeless" situation is also highlighted in Adler's recent book (1995). Nevertheless, we can overcome such a difficulty by using a complex scalar product and defining as the *appropriate momentum operator*

$$\mathbf{p} \equiv -\boldsymbol{\partial}|i \tag{13}$$

Note that the choice $\mathbf{p} \equiv -i\boldsymbol{\partial}$ still gives a self-adjoint operator with the standard commutation relations with the coordinates, but such an operator does not commute with the Hamiltonian, which will, in general, be a quaternionic quantity. Obviously, in order to write equations that are relativistically covariant, we must treat the space components and time in the same way, hence we are obliged to modify the standard "complex" equations by the following substitutions:

$$\iota\partial_t \to \partial_t|i \qquad \text{and} \qquad \iota\boldsymbol{\partial} \to \boldsymbol{\partial}|i$$

Thus, the four-momentum operator becomes $p^{\mu} \equiv \partial^{\mu}|i$.

3.1.2. *Doubling of Solutions*

The introduction of a complex projection for quaternionic scalar products gives an interesting *doubling* of solutions. Indeed, we observe that the dimensionality of a complete set of states for complex inner products $\langle\psi|\varphi\rangle_c$ is *twice* that of the quantum inner product $\langle\psi|\varphi\rangle$. Specifically, if $|\xi_l\rangle$ are a complete set of intermediate states for the quaternionic inner product, so that

$$\langle\psi|\psi\rangle = \sum_l \langle\psi|\xi_l\rangle\langle\xi_l|\varphi\rangle$$

$|\xi_l\rangle$ and $|\xi_l\, j\rangle$ form a complete set of states for the complex inner product

$$\begin{aligned}|\varphi\rangle &= \sum_l |\xi_l\rangle\langle\xi_l|\varphi\rangle_c + |\xi_l j\rangle\langle\xi_l j|\varphi\rangle_c \\ &= \sum_m |\chi_m\rangle\langle\chi_m|\varphi\rangle_c\end{aligned}$$

where $|\chi_m\rangle$ represent *complex* orthogonal states.

The choice of a complex geometry also justified the so-called "symplectic" complex representation of a quaternionic state

$$q = z + j\tilde{z}, \qquad z, \tilde{z} \in C(1, i) \subset \mathcal{H}$$

by the complex column matrix

$$q \leftrightarrow \begin{pmatrix} z \\ \tilde{z} \end{pmatrix} \qquad z, \tilde{z}: \qquad i \to \iota \tag{14}$$

3.2. Complexified QQM

As remarked at the beginning of this section, assuming complexified quaternionic amplitudes, we cannot give a satisfactory probability interpretation. Thus, we must work with quaternionic or complex scalar products.

The considerations taken in Section 3.1 can be immediately extended within complexified QQM. To define an appropriate momentum operator, we choose to adopt a complex geometry. Nevertheless, complex geometry represents an ambiguous term in complexified QQM. In fact, we have the following two possibilities:

(a) ι-complex (iota-complex) geometry, the binary mapping $\langle\Psi|\Phi\rangle_{\iota}$ of $V_{\mathcal{H}_c} \times V_{\mathcal{H}_c}$ into $\mathcal{C}(1, \iota)$,

$$\langle\Psi|\Phi\rangle_{\iota} = \frac{1 - i|i - j|j - k|k}{4} \langle\Psi|\Phi\rangle \tag{15}$$

(b) i-complex geometry, the binary mapping $\langle\Psi|\Phi\rangle_i$ of $V_{\mathcal{H}_c} \times V_{\mathcal{H}_c}$ into $C(1, i)$,

$$\langle\Psi|\Phi\rangle_i = \frac{1 - i|i}{2} \langle\Psi|\Phi\rangle \tag{16}$$

Consequently, we have to introduce *two* different momentum operators.

3.2.1. "ι-Complex" Momentum Operator

Recalling the commutation rules between the complex imaginary unit ι and the quaternionic imaginary units $\mathbf{h} \equiv (i, j, k)$

$$[\iota, \mathbf{h}] \equiv 0$$

we define, within complexified QQM with ι-complex geometry, the following momentum operator:

$$\mathbf{p} \equiv \iota\boldsymbol{\partial} \qquad (\equiv -\boldsymbol{\partial}|\iota) \tag{17}$$

This self-adjoint operator gives the standard commutation relations with the coordinates and obviously commutes with the quaternionic Hamiltonian.

3.2.2. "ι-Complex" Momentum Operator

Within complexified QQM with i-complex geometry, the choice of one of the three imaginary quaternionic units to locate the complex plane of projection for scalar products requires us to define the momentum operator by the right-action of the imaginary unit i. Thus, the appropriate definition of momentum operator, within complexified QQM with i-complex geometry, is practically that one given in equation (13), namely

$$\mathbf{p} \equiv -\boldsymbol{\partial}|i$$

3.2.3. Quadrupling of Solutions

We observe that both the ι-complex (iota-complex) and the i-complex geometry present a *quadrupling* of solutions. Specifically, if $|\xi_l\rangle$ are a complete set of intermediate states for the complexified quaternionic inner product, so that

$$\langle \Psi | \Phi \rangle = \sum_l \langle \Psi | \xi_l \rangle \langle \xi_l | \Phi \rangle$$

then $|\xi_l\rangle$, $|\xi_l i\rangle$, $|\xi_l j\rangle$, $|\xi_l k\rangle$ form a complete set of states for the ι-complex inner product

$$\begin{aligned} |\Phi\rangle &= \sum_l |\xi_l\rangle\langle\xi_l|\phi\rangle_\iota + |\xi_l i\rangle\langle\xi_l i|\Phi\rangle_\iota + |\xi_l j\rangle\langle\xi_l j|\phi\rangle_\iota + |\xi_l k\rangle\langle\xi_l k|\Phi\rangle_\iota \\ &= \sum_n |\eta_n\rangle\langle\eta_n|\Phi\rangle_\iota \qquad |\eta_n\rangle \quad \iota\text{-}complex \text{ orthogonal states} \end{aligned}$$

and $|\xi_l\rangle$, $|\xi_l \iota\rangle$, $|\xi_l j\rangle$, $|\xi_l \iota j\rangle$ form a complete set of states for the i-complex inner product

$$\begin{aligned} |\Phi\rangle &= \sum_l |\xi_l\rangle\langle\xi_l|\phi\rangle_i + |\xi_l \iota\rangle\langle\xi_l \iota|\Phi\rangle_i + |\xi_l j\rangle\langle\xi_l j|\phi\rangle_i + |\xi_l \iota j\rangle\langle\xi_l \iota j|\Phi\rangle_i \\ &= \sum_n |\zeta_n\rangle\langle\zeta_n|\Phi\rangle_i \qquad |\zeta_n\rangle \quad i\text{-}complex \text{ orthogonal states} \end{aligned}$$

This quadrupling of solutions will be the key in writing a one-dimensional (complexified) quaternionic Dirac equation.

4. EVEN-DIMENSIONAL TRANSLATION

Complex linear real quaternionic operators are characterized by four complex numbers and real quaternionic states by two *complex* orthogonal states. We also showed that ι-complex linear barred operators distinguish 16 complex parameters and complexified quaternionic states locate four ι-*complex* orthogonal states. This suggests a possible identification between 2 × 2 complex matrices and real quaternions and between 4 × 4 complex matrices and complexified quaternions.

4.1. Real Quaternionic Translation

The operator representation of i, j, and k consistent with the identification (14)

$$i \leftrightarrow \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix} = \iota\sigma_3, \qquad j \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\iota\sigma_2, \qquad k \leftrightarrow \begin{pmatrix} 0 & -\iota \\ -\iota & 0 \end{pmatrix} = -\iota\sigma_1 \tag{18}$$

has been known since the discovery of quaternions. It permits any quaternionic number or matrix to be translated into a complex matrix, *but not necessarily* *vice versa.* Eight real numbers are required to define the most general 2 × 2 complex matrix, but only four are needed to define the most general quaternion. In fact, since every (nonzero) quaternion has an inverse, only a subclass of invertible 2 × 2 complex matrices is identifiable with quaternions. Complex linear quaternionic operators complete the translation. The barred quaternionic imaginary unit

$$1|i \leftrightarrow \begin{pmatrix} \iota & 0 \\ 0 & \iota \end{pmatrix}$$

adds four additional degrees of freedom, obtained by matrix multiplication of the corresponding matrices,

$$1|i, \quad i|i, \quad j|i, \quad k|i$$

and so we have a set of rules for translating from any 2 × 2 complex matrices to complex linear barred operators and *vice versa* (Appendix B1). This opens new possibilities for quaternionic numbers; see, for example, the quaternionic version of the relativistic equations (Rotelli, 1989a; De Leo, 1995), the one-dimensional version of the Glashow group (De Leo and Rotelli, 1996a), the quaternionic Lagrangian formalism (De Leo and Rotelli, 1996b), noncommutative grand unification theories (De Leo, 1996c), and hyper-complex groups (De Leo, n.d.).

4.2. Complexified Quaternionic Translation

In analogy to the representation (14), we introduce for complexified quaternionic states

$$q_c = c_0 + ic_1 + jc_2 + kc_3, \qquad c_{0,1,2,3} \in \mathscr{C}(1, \iota)$$

the "symplectic" ι-complex representation by the following four-dimensional vector column:

$$q_c \leftrightarrow \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{19}$$

Repeating the same steps of the previous subsection, we find that the operator representation of the quaternionic imaginary units $\mathbf{h} \equiv (i, j, k)$ consistent with the above identification is

$$i \leftrightarrow \begin{pmatrix} -\iota\sigma_2 & 0 \\ 0 & -\iota\sigma_2 \end{pmatrix}, \qquad j \leftrightarrow \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad k \leftrightarrow \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \tag{20}$$

whereas the complex imaginary unit ι is identified by the matrix $\iota 1_{4\times4}$. Obviously, in order to complete the translation rules, we must also give the representation for the right-action of the quaternionic imaginary units, namely

$$1|i \leftrightarrow \begin{pmatrix} -\iota\sigma_2 & 0 \\ 0 & \iota\sigma_2 \end{pmatrix}, \quad 1|j \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad 1|k \leftrightarrow \begin{pmatrix} 0 & -\iota\sigma_2 \\ -\iota\sigma_2 & 0 \end{pmatrix} \tag{21}$$

Note that, as expected, $[\mathbf{h}, 1|\mathbf{h}] = 0$, and so in finding the matrix representation of the crossing left-right imaginary units we can choose left or right multiplication. For example,

$$\begin{aligned} k|i &\leftrightarrow \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} -\iota\sigma_2 & 0 \\ 0 & \iota\sigma_2 \end{pmatrix} \leftrightarrow k \times (1|i) \\ &\leftrightarrow \begin{pmatrix} -\iota\sigma_2 & 0 \\ 0 & \iota\sigma_2 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \leftrightarrow (1|i) \times k \end{aligned}$$

The complete set of rules for translating from any 4×4 complex matrices to ι-complex linear barred operators and vice versa is given in Appendix B2.

5. THE DIRAC EQUATION

We now have all the necessary tools to write down a quaternionic Dirac equation. In fact, we defined an appropriate momentum operator and gave the key for performing a quaternionic translation of 4×4 complex matrices by real and complexified quaternions. We then derive the quaternionic Dirac equation not from first principles, but simply by translating the standard complex equation. In Section 5.3 we also discuss the possibility of a "surprising" translation by i-complex linear (complexified) quaternionic operators.

In the following we shall perform quaternionic translations of the complex Dirac equation

$$\iota\gamma^\mu \partial_\mu \psi(x) = m\psi(x)$$

where the γ^μ-matrices have the standard representation (Itzykson and Zuber, 1985)

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{22}$$

Usually $\psi(x) \in C^4$ is a column spinor, but in Section 5.2 we shall regard it as a 2×2 complex matrix. This will appear clear once we achieve a one-dimensional (complexified) quaternionic formulation of the Dirac equation.

5.1. Real Quaternionic Representation

Noting that the imaginary unit which characterizes the momentum operator must appear on the right-hand of the wave function ($p^{\mu} \equiv \partial^{\mu}|i$), we write down the following real quaternionic Dirac equation:

$$\gamma^{\mu}\partial_{\mu}\psi(x)i = m\psi(x) \tag{23}$$

Now, $\psi(x) \in \mathcal{H}^2$ is a quaternionic column spinor. The two-dimensional quaternionic version of the standard 4×4 complex γ^{μ}-matrices is given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma} = \mathbf{q}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{24}$$

with

$$\mathbf{q} \equiv (k, j, -i)|i$$

Obviously, the standard results are soon obtained by translating the complex formulation. At first sight this is not the same as the quaternionic γ^{μ} set given in Rotelli (1989a) (except for γ^0); however, there exists a similarity transformation which transforms the above set into

$$\gamma^0_{\text{ref}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma}_{\text{ref}} = \mathbf{h}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\mathbf{h} \equiv (i, j, k)$. Explicitly,

$$\mathcal{S}\gamma^{\mu}\mathcal{S}^{-1} = \gamma^{\mu}_{\text{ref}}$$

with

$$\mathcal{S} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+j & 0 \\ 0 & (1+j)|i \end{pmatrix}$$

5.2. Complexified Quaternionic Representation

In virtue of our translation rules, we can obtain a one-dimensional complexified quaternionic Dirac equation. Due to its commutation properties, the imaginary unit $\boldsymbol{\iota}$, which characterizes the momentum operator, can now appear on the left-hand. So the one-dimensional Dirac equation reads

$$\boldsymbol{\iota}\gamma^{\mu}\partial_{\mu}\Psi(x) = m\Psi(x) \tag{25}$$

where $\Psi(x) \in \mathcal{H}_c$ is a (complexified) quaternionic spinor. The quaternionic representation for the standard γ^{μ}-matrices is soon obtained by translation (Appendix B2),

$$\gamma^0 = -i|i \qquad \text{and} \qquad \boldsymbol{\gamma} = -(k, \iota i|j, j) \tag{26}$$

Obviously, we can directly obtain a more elegant quaternionic representation for the γ^μ-matrices; for example,

$$\gamma^0 = i|i \qquad \text{and} \qquad \boldsymbol{\gamma} = \iota \mathbf{h}|j$$

satisfy the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

and

$$\gamma^{0\dagger} = \gamma^0, \qquad \boldsymbol{\gamma}^\dagger = -\boldsymbol{\gamma}$$

The possibility to write down a one-component quaternionic version of the Dirac equation gives an interesting "bonus." In fact, ι-complex linear barred operators can be reinterpreted by the left and right action of *simple* complexified quaternions, and *simple* complexified quaternions can be represented by 2×2 complex matrices

$$z_0 + j\tilde{z}_0 + \iota(z_1 + j\tilde{z}_1) \leftrightarrow \begin{pmatrix} z_0 + \iota z_1 & -\tilde{z}_0^* - \iota\tilde{z}_1^* \\ \tilde{z}_0 + \iota\tilde{z}_1 & z_0^* + \iota z_1^* \end{pmatrix}, \qquad z_{0,1}, \tilde{z}_{0,1}: \quad i \to \iota \tag{27}$$

where we use the identification (18) for the imaginary units i, j, k. The result is a two-dimensional complex representation of the Dirac equation. The explanation is simple once seen. In the standard theory the 4-dimensional complex freedom degrees of the spinor are represented by a four-dimensional vector column. Nevertheless, we have another possibility, namely to represent such 4 complex freedom degrees by a 2×2 complex matrix

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} \psi_a & \psi_b \\ \psi_c & \psi_d \end{pmatrix}$$

In both the cases the most general transformation which can be performed on our spinors will be characterized by 16 complex parameters. For a 4-dimensional vector column spinor, we obviously have the standard *left action* of 4×4 complex matrices, whereas for 2×2 complex matrix spinors, we find the possibility of *left/right action* of 2×2 complex matrices. The most general transformation on 2×2 complex matrix spinors will be

$$\mathcal{M}_0 + \mathcal{M}_1|\sigma_1 + \mathcal{M}_2|\sigma_2 + \mathcal{M}_3|\sigma_3$$

with $\mathcal{M}_{0,1,2,3}$ 2×2 complex matrices. Such transformation is again characterized by 16 complex parameters.

5.3. A "Surprising" Possibility

Working within complexified QQM with i-complex geometry, we also find a quadrupling of solutions and this suggests a correspondence between the standard column spinor $\psi \in C^4$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \qquad \psi_{1,2,3,4} \in \mathscr{C}(1, \iota)$$

and (complexified) quaternionic spinor $\Psi \in \mathcal{H}_c$

$$\Psi = \psi_1 + j\psi_2 + \iota(\psi_3 + j\psi_4), \qquad \psi_{1,2,3,4} \in C(1, i) \subset \mathcal{H}$$

Nevertheless, because of i-complex geometry, the most general transformation on the (complexified) quaternionic spinor ψ will be characterized by 8 i-complex parameters

$$\mathbb{O}_c^i \equiv q_c + p_c|i \qquad (q_c, p_c \in \mathcal{H}_c)$$

Due to missing complex parameters, we should have to meet difficulties in the formulation of the Dirac equation

$$\gamma^\mu \partial_\mu \Psi(x) i = m\Psi(x) \tag{28}$$

where the complex imaginary units have to appear on the right-hand side due to the momentum operator definition ($\mathbf{p} \equiv -\boldsymbol{\partial}|i$).

We now show that an interesting possibility surprisingly exists. Let us observe as follows. In finding the γ^μ-matrices satisfying the Dirac algebra, we have no problems with the γ-matrices, in fact we immediately find as suitable choice

$$\boldsymbol{\gamma} = \mathbf{h} \equiv (i, j, k), \qquad \{h^m, h^n\} = 2g^{mn} \qquad (m, n = 1, 2, 3), \qquad \mathbf{h}^\dagger = -\mathbf{h}$$

Nevertheless, we cannot find a quaternionic numbers which anticommutes with $\mathbf{h}$, and consequently we cannot give a (complexified) quaternionic representation for the γ^0-matrix. Working in complexified QQM with ι-complex geometry, the problem is overcome by using two "different" barred quaternionic imaginary units in representing γ^0 and γ. Explicitly,

$$\gamma^0 = i|i \qquad \text{and} \qquad \boldsymbol{\gamma} = \iota\mathbf{h}|j$$

Now we have only the barred imaginary unit $1|i$, and so this possibility is avoided.

However, we can have recourse to a "trick." The action of the standard γ^0-matrix on the complex spinor $\psi \in C^4$ is

$$\gamma^0\psi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ -\psi_3 \\ -\psi_4 \end{pmatrix}$$

In terms of complexified quaternions we have to find an operation which performs the following transformation:

$$\psi_1 + j\psi_2 + \iota(\psi_3 + j\psi_4) \rightarrow \psi_1 + j\psi_2 - \iota(\psi_3 + j\psi_4)$$

The solution is now obvious. The required operation is the $\bullet$-involution, $\Psi \rightarrow \Psi^\bullet$. Finally the Dirac equation

$$(\partial_t + \gamma^0\boldsymbol{\gamma} \cdot \boldsymbol{\partial})\Psi(x)i = m\gamma_0\Psi(x)$$

reads

$$(\partial_t + \iota\mathbf{h} \cdot \boldsymbol{\partial})\Psi(x)i = m\Psi^\bullet(x) \tag{29}$$

Since this equation is not obtained by simple translation, it requires particular study which will be developed in a forthcoming paper (De Leo and Rodrigues, n.d.).

6. ODD-DIMENSIONAL TRANSLATION

Up to now we have performed only even-dimensional translation. Odd complex representations of complex groups are excluded. Nonreducibility gives the existence of "anomalous" solutions. This simply followed from the complex geometry, which imposes the orthogonality of ψ, $j\psi$ within real QQM and of ψ, $i\psi$, $j\psi$, $k\psi$ within complexified QQM. This appears to exclude a complete translation between standard (complex) QM and QQM, which is good or bad news, according to one's point of view.

There exists a trivial way of bypassing this problem by eliminating the anomalous solutions, assuming that they are "nonphysical" solutions (Rotelli, 1989b). *This is not the correct interpretation* (De Leo and Rotelli, 1992).

Odd-dimensional complex representations will be reducible with barred quaternions thanks to the overlapping feature described below. Hence, the problem of having a reducible vector space for a nonreducible matrix representation will be *partially* eliminated.

6.1. "Overlapping" Translation

In this subsection we shall describe the "overlapping" technique by illustrating the situation for spin 1. The general rules for any odd-dimensional matrix can be then extracted by a simple "trick".

The three "complex" anti-hermitian generators of spin 1, A_m ($m = 1, 2, 3$), are, in standard form,

$$A_1 = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3 = -i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{30}$$

These have normal/anomalous solutions

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} j \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}$$

Thus, each eigenvalue is degenerate and the vector space represented by the column matrices is reducible to two three-dimensional subspaces. The conventional form of reductions of the 3×3 generators is not possible because it would require the division of the 3×3 matrices A_m into two distinct blocks, one of 2×2 (quaternionic in general, and this is possible) and one of 1×1, and this can be explicitly excluded. Hermitian generators of $SU(2)$ exist, they are $i|i$, $j|i$, and $k|i$. However, these correspond to spin-1/2 eigenvalues and not spin 1. It is easy to convince oneself that no one-dimensional spin-1 representations exist.

This is an example of the reduction problem mentioned previously. Now we shall show explicitly that there exists a generalized quaternionic similarity matrix S ($S^\dagger = S^{-1}$) such that the A_m are reduced to two overlapping 2×2 block forms so that one element, the (2,2)-element, is common to both blocks. However, if this element if not identically zero, it is always a composite of two terms, one of which annihilates one of the corresponding eigenvectors. Thus the two blocks may be unlocked and studied separately.

Explicitly an S matrix such as that described above is given by

$$S = 2\begin{pmatrix} 1 - i|i & j(1 - i|i) & 0 \\ 0 & 0 & \frac{1}{2} \\ (1 + i|i) & -j(1 + i|i) & 0 \end{pmatrix} \tag{31}$$

The transformed generators $\tilde{A}_m = SA_mS^\dagger$ are then given by

$$\tilde{A}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} k & ka & 0 \\ kd & 0 & -ka \\ 0 & -kd & -k \end{pmatrix}$$

$$\tilde{A}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} j & -ja & 0 \\ -jd & 0 & ja \\ 0 & jd & -j \end{pmatrix}$$

$$\tilde{A}_3 = -i\begin{pmatrix} a & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

with $2a = 1 - i|i$ and $2d = 1 + i|i$. In $\tilde{A}_3$ the (2,2)-element can be written conveniently as $i(a + d)$, i.e., containing a sum of projection operators. The corresponding transformed state vectors with eigenvalues $+1$, 0, -1 are, respectively,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} j \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ j \\ 0 \end{pmatrix}$$

We observe that the first triplet consists of vectors of the form

$$\begin{pmatrix} q \\ z \\ 0 \end{pmatrix}$$

while the second triplet consists of vectors of the form

$$\begin{pmatrix} 0 \\ jz \\ q \end{pmatrix}$$

Naturally the two triplets remain orthogonal with a complex geometry and furthermore the separate sets of reduced 2×2 quaternionic blocks do not perform any rotation upon one or other set of triplets. Actually the two sets of reduced 2×2 generators are connected by a similarity transformation and thus are in turn equivalent. Explicitly the sets of 2×2 reduced quaternionic representations are

$$\tilde{B}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} k & ka \\ kd & 0 \end{pmatrix}, \quad \tilde{B}_2 = \frac{-1}{\sqrt{2}}\begin{pmatrix} -j & ja \\ jd & 0 \end{pmatrix}, \quad \tilde{B}_3 = -ia\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{C}_1 = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 & ka \\ kd & k \end{pmatrix}, \quad \tilde{C}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & ja \\ jd & -j \end{pmatrix}, \quad \tilde{C}_3 = -id\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The corresponding state vectors with eigenvalues $+1, 0, -1$, respectively, are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ j \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} j \\ 0 \end{pmatrix}$$

It is of course natural to ask what the translation of these reduced 2×2 generators to complex form yields. The answer, which seems obvious *a posteriori,* is the 4×4 complex generators of $SU(2)$ *reducible* to spin-1 $\oplus$ spin-0. Not, of course, the irreducible 4×4 representations, which would correspond to spin 3/2. Because of our derivation we would be tempted to identify the spin-0 element as a member of an independent triplet, but this has no physical foundation. What is significant is that the reduction is not perfect in the sense that it brings along a *scalar* partner.

These results lead to the following consequences: One is a mechanical (automatic) way of reducing *any* odd-dimensional (otherwise irreducible) complex matrix with quaternions into *overlapping block* structure. The second is the physical significance of this procedure. For the first, we propose to add an extra row and column of zeros to our matrix, thus making it become an even matrix and then applying the *translation* of this complex matrix to quaternions. This is a formal *trick* since we began with a complex odd-dimensional matrix operating upon a quaternionic space, i.e., considered as a quaternionic matrix without need of translation and with only a question about its reducibility. Nevertheless, this trick always yields one or other of the overlapping block forms.

We wish to emphasize that the previous odd translation is possible, but it is *not* obligatory. In fact, "complex" groups or dynamical equations can be allowed within quaternionic physical theory. In such a case we must find a physical interpretation for our "anomalous" solutions.

6.2. "Anomalous" Solutions and Their Physical Interpretation

In this subsection we give some possible interpretations of "anomalous" solutions.

6.2.1. Real QQM

We shall treat the color group $SU(3)$ and the standard Klein–Gordon equation. In discussing the color group we find three possibilities (De Leo, 1996c).

1. The "complex" group $SU(3\ C_{right})$. In this case, we can interpret the complex solutions as u-states and the anomalous solutions as jd-states. It is straightforward to verify that the quaternionic electroweak group $U(1,q)$ does not mix the quark freedom degrees *red, green, blue* (De Leo, 1997).

2. The "complex" group $SU(3,C_{left})$. We also have a doubling of states, but in this case the complex solutions transform like 3, whereas the j-complex solutions like 3* ($ij = ji^*$). So we have additional multiplets. The minimal grand unification group $SU(5)$ (Georgi and Glashow, 1974) will have (in its quaternionic version) the following additional multiplets $j5^*$ and $j10^*$. We know that a single unification point cannot be obtained within minimal (nonsupersymmetric) $SU(5)$. In the "quaternionic" version of $SU(5)$ additional multiplets of quarks and leptons naturally appear and allow the right unification properties (Amaldi *et al.*, 1992).

3. The "quaternionic" group $SU(2,\ q + p|i)$. In this case we start with the quaternionic counterpart of $SU(4)$ and break down a two-dimensional quaternionic color group, which represents the odd-dimensional quaternionic translation of the group $SU(3)$. In this case we must obviously admit a *fourth* color. Following the idea of Pati and Salam (1973), we can put the fermions of the first generation in the multiplets

$$\begin{pmatrix} u_r + ju_g \\ u_b + j\nu_w \end{pmatrix}, \qquad \begin{pmatrix} d_r + jd_g \\ d_b + je_w \end{pmatrix}$$

Let us direct our attention to the "anomalous" scalars of the Klein–Gordon equation (De Leon and Rotelli, 1992).

The "Quaternionic" KG equation. Since the only fundamental scalar could be the Higgs boson, in order to interpret the anomalous scalars we believe it to be natural to concentrate our attention on the Higgs sector (De Leo and Rotelli, 1995) of the electroweak theory. In the quaternionic version of the Salam–Weinberg model (Weinberg, 1967; Salam, 1968), the anomalous scalars of the Klein–Gordon equation are fundamental to obtain the necessary "four-complex" freedom degrees of Higgs scalars before symmetry breaking,

$$\Phi = \varphi + j\tilde{\varphi}, \qquad \varphi,\ \tilde{\varphi} \quad \text{complex scalar field}$$

It is also immediate to observe that the minimal four Higgs of the unbroken electroweak theory naturally determine the quaternionic invariance group which corresponds to the Glashow group (Glashow, 1961)

$$U(1, \mathbf{h}) | U(1, i) \tag{32}$$

6.2.2. Complexified QQM

In this subsection we direct our attention to the quadrupling of solutions of the "complexified quaternionic" Klein–Gordon equation.

The "quaternionic" DKP equation. The (complex) Kemmer equation (Klemmer, 1938)is

$$\beta^{\mu}\partial_{\mu}\varphi = m\varphi \tag{33}$$

(where the i-factor of the momentum operator has been absorbed into β^{μ}) with the β^{μ}-matrices satisfying the Duffin–Kemmer–Petiau (DKP) condition (Petiau, 1936; Géhéniau, 1938; Duffin, 1938)

$$\beta^{\mu}\beta^{\nu}\beta^{\lambda} + \beta^{\lambda}\beta^{\nu}\beta^{\mu} = -g^{\mu\nu}\beta^{\lambda} - g^{\lambda\nu}\beta^{\mu} \tag{34}$$

The DKP equation has spin content 0 and 1. Equation (34) implies that β^{μ}'s are not invertible, so that this equation cannot be written in the Dirac form

$$\partial_t \varphi i = H\varphi$$

It requires a 16-dimensional representation for the β^{μ}-matrices. Thus, by translating the "complex" β^{μ}-matrices, we can immediately write down a 4-dimensional "quaternionic" DKP equation. In such a case we have no anomalous solution, because of the reduced matrix dimensions of the quaternionic DKP equation (the same situation appeared in the Dirac equation).

Historically, the loss of the interest in the DKP equation stems from the equivalence of the DKP equation to the KG and Proca equations (Fischbach *et al.,* 1972; Krajcik and Nieto, 1974), in addition to the great algebraic complexity of the DKP formulation. In the quaternionic world the KG equation is equivalent to *four* DKP equations (16 complex matrices split into four 4-dimensional quaternionic blocks). It is straightforward to show that these four DKP equations can be taken back to four *new* KG equations (without anomalous solutions), namely

$$[1 - i|i - j|j - k|k](\partial^{\mu}\partial_{\mu} + m^2)\Phi = 0$$

$$[1 - i|i + j|j + k|k](\partial^{\mu}\partial_{\mu} + m^2)\Phi = 0$$

$$[1 + i|i - j|j + k|k](\partial^{\mu}\partial_{\mu} + m^2)\Phi = 0$$

$$[1 + i|i + j|j - k|k](\partial^{\mu}\partial_{\mu} + m^2)\Phi = 0$$

Table I.

CQM	Real QQM	Complexified QQM
$U(1)$ Electromagnetic group	$U(1, i) \otimes U(1, \mathbf{h})$ "Standard" electroweak group	$U(1, \iota) \otimes U(1, \mathbf{h}) \otimes U(1, \mathbf{h})$ "Left right" electroweak group

Note that we are *not* obliged to kill the anomalous solutions. The correct equations and the corresponding Lagrangians are in practice determined only when the number of particles in the theory is fixed.

6.2.3. *Quaternionic Phases*

We conclude this section with some considerations on the symmetry operations. A symmetry operation $\mathcal{S}$ of a system described by $|\psi$ is a mapping of $|\psi\rangle$ into $|\psi'\rangle$ which preserves all transition probabilities

$$\mathcal{S}|\psi\rangle = |\psi'\rangle$$

$$|\langle\varphi'|\psi'\rangle|^2 = |\langle\varphi|\psi\rangle|^2$$

In QQM with complex geometry quaternionic/complex phases

$$e^{\boldsymbol{\alpha}\cdot\mathbf{h}}|e^{i\gamma} \qquad \text{(real quaternions)}$$
$$e^{\iota\gamma}e^{\boldsymbol{\alpha}\cdot\mathbf{h}}|e^{\boldsymbol{\beta}\cdot\mathbf{h}} \qquad \text{(complexified quaternions)}$$

appear. We can immediately prove that the previous transformations represent an invariance of $\langle\psi|\varphi\rangle_i$ and $\langle\Psi|\Phi\rangle_\mathrm{i}$,

$$\langle\psi|\varphi\rangle_i = e^{-i\gamma}\langle\psi|\varphi\rangle_i\, e^{i\gamma} = \langle\psi|\varphi\rangle_i$$

$$\langle\Psi|\Phi\rangle_\iota = e^{-\boldsymbol{\beta}\cdot\mathbf{h}}\langle\Psi|\Phi\rangle_\iota\, e^{\boldsymbol{\beta}\cdot\mathbf{h}} = \langle\Psi|\Phi\rangle_\iota$$

Thus, we could propose the identification given in Table I for the "minimal" invariance groups which appear in CQM and real/complexified QQM

7. CONCLUSIONS

In this paper we showed a set of rules for passing back and forth between standard QM and real/complexified QQM. It is important to recall that the possibility of rewriting particle physics theories in quaternionic form is a nontrivial objective; in fact the noncommutative nature of quaternions alters the conventional approach. A fundamental ingredient in performing quaternionic versions of standard theories is surely represented by the choice of a "complex geometry". It allows the definition of an appropriate momentum operator by the introduction of the so-called barred operators.

As example of quaternionic translation of standard theories, we examined the Dirac equation. In a pure translation nothing can be predicted which is not already in the original theory, nevertheless some assumptions may become more "natural," some calculations may be more rapid, and some "new" (hidden) results may appear in the translated version for the first time. Let us briefly explain this last consideration. The Dirac equation in the complex world is expressed by 4 × 4 complex matrices and the spinor ψ is represented by a 4-dimensional (noninvertible) vector column. If we use real quaternions, we halve the dimensions of our γ^{μ}-matrices and the spinor ψ. Nevertheless, nothing new appears. Going to complexified quaternions, we were able to write a *one-component* Dirac equation and as consequence of this, we reinterpreted the Dirac spinor ψ by a complexified quaternion Ψ. In this formalism we find two "bonuses":

1. Complexified quaternions can be represented by 2 × 2 complex matrices and this suggests the possibility to write down a 2 × 2 complex Dirac equation, showing that the Pauli algebra is sufficient to reproduce the standard Dirac results.

2. Complexified quaternionic spinors Ψ are now invertible

$$\Psi^{-1} \equiv \frac{\Psi^{\star}}{\Psi\Psi^{\star}}, \qquad \Psi\Psi^{\star} \in \mathcal{C}(1, \iota)$$

We recall that the representation of a Dirac spinor by a generic (invertible) element of the Pauli algebra has given new insights in Dirac's theory of the electron (Hestenes, 1966, 1967, 1975, 1991; Rodrigues and Capelas de Oliveira, 1990; Zeni and Rodrigues, 1992).

In the quaternionic world we also find embarrassment with purely complex odd-dimensional matrix representations of a group acting upon a quaternionic space. The space representation is reducible, but the generator representation is not. We showed that odd-dimensional complex matrices are reducible if we allow for overlapping block structures. This will be possible in all situations, so these "translations" are only partial. We wish to emphasizes that we are *not* obliged to kill the anomalous solutions. They can be reinterpreted within unification theories. Besides, the presence of such anomalous solutions in the (standard) KG equation suggests *new* invariance groups. We propose the following analogy:

Complex	→	Real quaternions	
Electroagnetism	→	Standard electroweak	
		→	Complexified quaternions
		→	Left/right electroweak symmetry

However, one cannot elude the impression that quaternions invite an *even number of particle states.*

Before concluding this paper, let us address some considerations on the complexified quaternionic version of the Dirac equation given in Section 5.3. If right, it could represent the "natural" representation for the Dirac equation. Why?

1. It is immediately translated into a two-dimensional complex Dirac equation, where we immediately recognize the element of the Pauli algebra.

2. Its solutions, obviously of the form $\Psi = U\, e^{-ipx}$ (remember that the momentum operator is defined by the right action of the quaternionic imaginary unit i), in the case of rest frame, are

$$\Psi_+ \sim (1, j)e^{-imt} \quad \text{and} \quad \Psi_- \sim \iota(1, j)e^{+imt}$$

Thus, we can give an interesting geometric interpretation of the Dirac solutions (at least for particle with $\mathbf{p} = 0$). Solutions of negative energy are obtained by these of positive energy with rotations of $\pi/2$ in the complex plane $(1, \iota)$.

3. The • involution can be immediately related to space inversion,

$$\Psi(\mathbf{x}, t) \rightarrow \Psi^{\bullet}(-\mathbf{x}, t) \quad \text{leaves invariant the Dirac equation}$$

and this suggests a possible connection between the */† conjugations (which represent antiautomorphisms) and T/C operations. The CPT theorem should be a natural consequence of the three conjugations which appear in the complexified quaternionic field.

4. The nonrelativistic limit, obtained by killing the the negative-energy solutions, should be characterized by pure quaternionic solutions. In this case, the nonrelativistic limit should represent an intrinsic property of (complexified) quaternionic solutions, in contrast to the complex and real quaternionic cases, where the nonrelativistic limit gives a dimensionally reduced Scröd-inger–Pauli equation.

5. We find the interesting situation given in Table II in discussing the "complex" Schrödinger equation.

These last considerations obviously represent speculations which need to be deeply investigated. We shall develop a complete and clear study of the complexified quaternionic Dirac equation (within QQM with i-complex geometry) in a forthcoming paper (De Leo and Rodrigues, n.d.) For the moment, we hope to have simplified and clarified the mathematical language used in the quaternionic world. The use of quaternions in physical theories

Table II.

Complex	Real quaternions	Complexified quaternions
One solution	Two solutions Spin ↑ ↓, $E > 0$	Four solutions Spin ↑ ↓, $E > 0$, $E < 0$

is surely possible, and in some cases it can provide additional predictions and new geometric interpretations.

APPENDIX A. HISTORICAL REVIEW

In this appendix we give a little of the history of quaternions. The history of this subject is dominated by the *extraordinary contrast* of two personalities (Altmann, 1986): Sir William Rovan Hamilton and Olinde Rodrigues.

A1. Hamilton's Discovery

On 16 October 1843 Hamilton was walking along the Royal Canal with his wife to preside at a council meeting of the Royal Irish Academy. Although his wife talked to him now and again, Hamilton hardly listened, for the discovery of the quaternions, the first noncommutative algebra to be studied, was taking shape in his mind.

Hamilton knew the geometrical Argand diagram definition of the complex numbers, but preferred to define complex numbers as pairs (a,b) of real numbers. He had set himself the task of finding how triplets (a,b,c) of real numbers would multiply to give a 3-dimensional analogue. After a number of failed attempts, inspiration struck that day as he walked by the Royal Canal.

> *And here there dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triples . . . An electric circuit seemed to close, and a spark flashed forth.*

He could not resist the impulse to carve the formulas for quaternions in the stone of Brougham Bridge as he and his wife passed it. Hamilton felt this discovery would revolutionize mathematical physics and he spent the rest of his life working on quaternions.

Letter Describing the Discovery of Quaternions

From Sir W. R. Hamilton to Rev. Archibald H. Hamilton
Letter dated August 5, 1865

My dear Archibald,

(1) I had been wishing for an occasion of corresponding a little with you on quaternions and your note of yesterday, received this morning, that you "have been reflecting on several points connected with them" (quaternions), "particularly on the *Multiplication of vectors*".

(2) No more important, or indeed fundamental question, in the whole Theory of Quaternions; can be proposed than that which thus inquires *What is such Multiplication?* What are its Rules, its Objects, its Results? What

Analogies exist between it and other *Operations* which have received the same general *Name?* And finally, what is (if any) its Utility?

(3) If I may be allowed to speak of *myself* in connexion with the subject, I might do so in a way which would bring *you* in, by referring to an *ante-quaternionic,* time, when you were a mere *child,* but had caught from me the conception of a *Vector,* as represented by a *Triplet:* and indeed I happen to be able to put the finger of memory upon the year and month—October, 1843—when having recently returned from visits to Cork and Parsonstown, connected with a meeting of the British Association the desire to discover the laws of the multiplication referred to regained with me a certain strength and earnestness which had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, "Well, Papa, can you *multiply* triplets"? Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only *add* and *subtract* them".

(4) But on the 16th day of the same month—which happened to be a Monday, and a Council day of the Royal Irish Academy—I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven, and although she talked with me now and then, yet an *under-current* of thought was going on in my mind, which gave at last a *result,* whereof it is not too much to say that I felt *at once* the importance. An *electric* circuit seemed to *close,* and a spark flashed forth, the herald (as I *foresaw, immediately*) of many long years to come of definitely directed thought and work, by *myself* if spared, and at all events on the part of *others,* if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse—unphilosophical as it may have been—to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k, namely

$$i^2 = j^2 = k^2 = ijk = -1$$

which contains the *Solution* of the *Problem,* but of course, as an inscription, has long since mouldered away. A more durable notice remains, however, on the Council Books of the Academy for that day (October 16th, 1843), which records the fact, that I then asked for and obtained leave to read a Paper on *Quaternions,* at the *First General Meeting* of the session: which reading took place accordingly, on Monday the 13th of the November following. With this *quaternion of paragraphs* I close this letter, but I hope to follow it up very shortly with another.

Your affectionate father,
William Rowan Hamilton

A2. Rodrigues' Discovery

Rotations are parametrized by means of the well-known Euler angles. A different approach is possible, and was introduced by Olinde Rodrigues in 1840. The rotation operators are obtained in this approach by an entirely geometric method, which not only provides a much better insight into their nature, but also leads most naturally to the parametrization of rotations by parameters that coincide with quaternions. These parameters provide an algebra that permits the multiplication of rotations in a very simple way.

Having solved the geometric problem as regards the product of two rotations, Rodrigues (1840) gives closed formulas for determining the resultant angle and axis of rotation. In order to do this, he parametrizes a rotation with four parameters. If α is the angle of rotation and $\hat{u} \equiv (u_x, u_y, u_z)$ the unitary vector denoting the axis of rotation, his parameters are

$$\cos\left(\frac{\alpha}{2}\right), \qquad u_x \sin\left(\frac{\alpha}{2}\right), \qquad u_y \sin\left(\frac{\alpha}{2}\right), \qquad u_z \sin\left(\frac{\alpha}{2}\right)$$

If they are used into a quaternion, taking the place of its real component, then the formula for the multiplication of rotations that Rodrigues provides is precisely Hamilton's multiplication rule for quaternions.

An elegant and clear discussion of quaternionic treatment of rotations is given in Altmann (1986).

APPENDIX B. EVEN-DIMENSIONAL TRANSLATION RULES

With the rules given in Section 4 we can translate any quaternionic operator into an equivalent even-dimensional complex matrix and *vice versa.*

B1. Real Quaternions

For the lowest order operator we have

$$z_0 + j\tilde{z}_0 + (z_1 + j\tilde{z}_1)|i \rightarrow \begin{pmatrix} z_0 + \iota z_1 & -\tilde{z}_0^* - \iota\tilde{z}_1^* \\ \tilde{z}_0 + \iota\tilde{z}_1 & z_0^* + \iota z_1^* \end{pmatrix}, \qquad z_{0,1},\ \tilde{z}_{0,1}: \quad i \rightarrow \iota \tag{35}$$

Equivalently a generic 2×2 complex matrix is given by

$$2\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow a + d^* + j(c - b^*) + \left(\frac{a - d}{i} + j\,\frac{c + b^*}{i}\right)|i,$$

$$a, b, c, d: \quad i \rightarrow \iota \tag{36}$$

B2. Complexified Quaternions

For the lowest order operator

$$q_c + p_c|i + r_c|j + s_c|k \equiv c_0^q + \mathbf{h}\cdot\mathbf{c}^q + (c_0^p + \mathbf{h}\cdot\mathbf{c}^p)|i + (c_0^r + \mathbf{h}\cdot\mathbf{c}^r)|j + (c_0^s + \mathbf{h}\cdot\mathbf{c}^s)|k$$

where $c_0^{q,p,r,s}$, $\mathbf{c}^{q,p,r,s} \in \mathscr{C}(1, \iota)$, we find the following 4×4 complex matrix representation:

$$\begin{pmatrix} 0^+1^-2^-3^- & 1^-0^-3^-2^+ & 2^-3^+0^-1^- & 3^-2^-1^+0^- \\ 1^+0^+3^-2^+ & 0^+1^-2^+3^+ & 3^-2^-1^-0^+ & 2^+3^-0^-1^- \\ 2^+3^+0^+1^- & 3^+2^-1^-0^- & 0^+1^+2^-3^+ & 1^-0^+3^-2^- \\ 3^+2^-1^+0^+ & 2^-3^+0^+1^- & 1^+0^+3^-2^- & 0^+1^-2^+3^- \end{pmatrix} \tag{37}$$

where we use the compact notation

$$0^+1^-2^-3^- \equiv +c_0^q - c_1^p - c_2^r - c_3^s$$

$$2^+3^+0^+1^- \equiv +c_2^q + c_3^p + c_0^r - c_1^s$$

$$\vdots$$

Equivalently a generic 4×4 complex matrix

$$4\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix}$$

is translated into

$$\begin{aligned} &a^+b^+c^+d^+ + ib^-a^+d^-c^+ + jc^-d^+a^+b^- + kd^-c^-b^+a^+ \\ &+ (b^-a^+d^+c^- + ia^-b^-c^+d^+ + jd^-c^-b^-a^- + kc^+d^-a^+b^-)|i \\ &+ (c^-d^-a^+b^+ + id^+c^-b^-a^+ + ja^-b^+c^-d^+ + kb^-a^-d^-c^-)|j \\ &+ (d^-c^+b^-a^+ + ic^-d^-a^-b^- + jb^+a^+d^-c^- + ka^-b^+c^+d^-)|k \end{aligned} \tag{38}$$

with

$$a^+b^+c^+d^+ \equiv +a_1 + b_2 + c_3 + d_4$$

$$c^-d^-a^+b^+ \equiv -c_1 - d_2 + a_3 + b_4$$

We now give sample examples. In the matrix

$$\gamma^0 = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix}$$

the only nonzero coefficients are $a_1 = b_2 = -c_3 = -d_4 = \frac{1}{4}$, and so the complexified quaternionic version is characterized by $a^+ b^+ c^- d^- = 1$, namely

$$\gamma^0 = ia^-b^-c^+d^+|i = -ia^+b^+c^-d^-|i \equiv -i|i \tag{39}$$

In the same way we can obtain the translation for the complex matrices

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$$

In this case we have

$$d^+c^+b^-a^- = 1, \qquad d^-c^+b^+a^- = i, \qquad c^+d^-a^-b^+ = 1$$

and so the complexified quaternionic version of the γ matrices is given by

$$\boldsymbol{\gamma} = (kd^-c^-b^+a^+,\ id^+c^-b^-a^+|j, jc^-d^+a^+b^-) \equiv -(k,\ i|j, j) \tag{40}$$

ACKNOWLEDGMENTS

One of the authors (S.dL.) gratefully acknowledges the Italian CNR and the Brazilian FINEP for financial support and the Direction of the IMECC for warm hospitality during a stay at the Instituto de Matemática, Estatística e Computação Cientifica (UNICAMP), where this paper was prepared.

REFERENCES

Adler, S. L. (1994). *Physics Letters,* **332B,** 358.
Adler, S. L. (1995). *Quaternionic Quantum Mechanics and Quantum Fields,* Oxford University Press, Oxford.
Albert, A. A. (1947). *Annals of Mathematics,* **48,** 495.
Altmann, S. L. (1986). *Rotations, Quaternions, and Double Groups,* Clarendon Press, Oxford.
Amaldi, U., de Boer, W., Frampton, P. H., *et al.* (1992). *Physics Letters,* **281B,** 374.
De Leo, S. (1995). *Progress of Theoretical Physics,* **94,** 1109.
De Leo, S. (1996a). *International Journal of Modern Physics A,* **11,** 3973.
De Leo, S. (1996b). *Journal of Mathematical Physics,* **37,** 2955.
De Leo, S. (1996c). *International Journal of Theoretical Physics,* **35,** 1821.
De Leo, S. (1997). *International Journal of Theoretical Physics,* **36,** 1165.
De Leo, S. (n.d.). Hypercomplex group theory, *Journal of Mathematical Physics,* submitted [physics/9703033].
De Leo, S., and Rodrigues, W. A. (n.d.). Quaternionic Dirac equation, in preparation.
De Leo, S., and Rotelli, P. (1992). *Physical Review D,* **45,** 575.
De Leo, S., and Rotelli, P. (1994). *Progress of Theoretical Physics,* **92,** 917.
De Leo, S., and Rotelli, P. (1995). *International Journal of Modern Physics A,* **10,** 4359.
De Leo, S., and Rotelli, P. (1996a). *Journal of Physics G,* **22,** 1137.
De Leo, S., and Rotelli, P. (1996b). *Modern Physics Letters A,* **11,** 357.
Duffin, R. J. (1938). *Physical Review,* **54,** 1114.

Finkelstein, D., Jauch, J. M., Schiminovich, S., Speiser, D. (1962). *Journal of Mathematical Physics*, **3**, 207.

Finkelstein, D., Jauch, J. M., and Speiser, D. (1963a). *Journal of Mathematical Physics*, **4**, 136.

Finkelstein, D., Jauch, J. M., Schiminovich, S., and Speiser, D. (1963b). *Journal of Mathematical Physics*, **4**, 788.

Finkelstein, D., Jauch, J. M., and Speiser, D. (1979). *Notes on quaternion quantum mechanics, in Logico-Algebraic Approach to Quantum Mechanics II*, C. A. Hooker, ed., Reidel, Dordrecht, pp. 367–421.

Fishbach, E., Nieto, M. M., and Scott, C. K. (1972). *Progress of Theoretical Physics*, **48**, 574.

Géhéniau, J. (1938). *Academie Royale de Belgique. Memoires de la Classe des Sciences. Collection in -8*, **18**, 1.

Georgi, H., and Glashow, S. L. (1974). *Physical Review Letters*, **32**, 438.

Glashow, S. L. (1961). *Nuclear Physics*, **22**, 579.

Hamilton, W. R. (1943). Quaternion Century Celebration, *Proceedings of the Royal Irish Academy, Section A*, **50**, 11.

Hamilton, W. R. (1969). *Elements of Quaternions*, Chelsea, New York.

Harari, H. (1979). *Physics Letters*, **86B**, 83.

Hestenes, D. (1966). *Space-Time Algebra*, Gordon and Breach, New York.

Hestenes, D. (1967). *Journal of Mathematical Physics*, **8**, 789.

Hestenes, D. (1975). *Journal of Mathematical Physics*, **16**, 556.

Hestenes, D. (1991). *Foundations of Physics*, **20**, 1213.

Horwitz, L. P., and Biedenharn, L. C. (1984). *Annals of Physics*, **157**, 432.

Itzykson, S., Zuber, J. B. (1985). *Quantum Field Theory*, McGraw-Hill, New York.

Kemmer, N. (1938). *Proceedings of the Royal Society*, **166**, 127.

Krajcik, R. A., and Nieto, M. M. (1974). *Physical Review D*, **10**, 4049.

Pati, J. C., and Salam, A. (1973). *Physical Review D*, **8**, 1240.

Petiau, G. (1936). *Academie Royale de Belgique. Memoires de la Classe des Sciences. Collection in -8*, **16**, 2.

Rembieliński, J. (1978). *Journal of Physics A*, **11**, 2323.

Rodrigues, O. (1840). *Journal de Mathémetiques Pures et Appliquées*, **5**, 380.

Rodrigues, W. A., Jr., and Capelas de Oliveira, E. (1990). *International Journal of Theoretical Physics*, **29**, 397.

Rotelli, P. (1989a). *Modern Physics Letters A*, **4**, 933.

Rotelli, P. (1989b). *Modern Physics Letters A*, **4**, 1763.

Salam, A. (1968). In *Proceedings of the 8th Nobel Symposium, Weak and Electromagnetic Interactions*, N. Svartholm, ed., Almqvist & Wiksell, Stockholm, and Wiley, New York, p. 367.

Shupe, M. A. (1979). *Physics Letters*, **86B**, 87.

Weinberg, S. (1967). *Physical Review Letters*, **19**, 1264.

Zeni, J. R., and Rodrigues, W. A., Jr. (1992). *International Journal of Modern Physics A*, **7**, 1793.